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SUFFICIENT CONDITIONS FOR THE SOLVABILITY AND SUPERSOLVABILITY IN FINITE GROUPS

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Let r be a natural number. A subgroup H of a finite group G is called an r-minimal subgroup of G if there is a chain of subgroups $1=H_0 \le H_1 \le \cdots \le H_r = H$ with each H_i is maximal in H_{i+1}. Naturally an r-minimal subgroup H of G can be also an m-minimal subgroup for $m \ne r$. In [1], N.S. Narasimha and W.E. Deskins said that a group G is a PN-r group if each r-minimal subgroup of G is normal. They showed that PN-r groups are solvable of fitting length at most r for $r=2$ and 3. They also obtained results about PN-4 groups.

Let G be a finite group of even order. Then by definition

$$\mathscr{H}_r = \{H<G \mid H \text{ is of even order and } |H| \text{ is the product of } r \text{ primes not necessarily distinct}\}.$$

We say that G is an $\mathscr{H}_r$-N group if each element of $\mathscr{H}_r$ is normal in G.

Let G be a finite group of odd order. Let p be the smallest prime in $\pi(G)$. Then by definition

$$\mathscr{K}_2 = \{H<G \mid p \text{ divides } |H| \text{ and } |H| \text{ is the product of two primes not necessarily distinct}\}.$$

A group G is a $\mathscr{K}_2$-N group if each element of $\mathscr{K}_2$ is normal in G.

Our notation is standard and is taken mainly from [2]. All groups considered are assumed to be finite.

The object of this paper is to prove the following theorems:

Theorem 1. *Suppose that each subgroup of order p^2 is normal in G for every prime divisor p of $|G|$ except perhaps the largest. Then either G possesses a Sylow tower or A_4 is involved in G.*

G possesses a Sylow tower, that is to say there is a series $1=G_0<G_1<\cdots<G_n=G$

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of normal subgroups of G such that for each $i = i, 2, \dots, n$, G_i/G_{i-1} is isomorphic to a p_i-Sylow subgroup of G, where $p_1, p_2, \dots, p_n$ are the distinct prime divisors of $|G|$ and $p_1 > p_2 > \cdots > p_n$. A familiar consequence of the supersolvability of G is that G possesses a Sylow tower [4, p. 716, VI.9.1].

For the proof of Theorem 1 we need the following definition: A group G is called a (p,q)-group if:

(a) The order of G involves only the prime factors p and q.

(b) G is not nilpotent and all its proper subgroups are nilpotent.

(c) The derived group G' is the p-Sylow subgroup of G.

We refer the reader to [4, p. 281, Theorem 5.2] for the standard prop· ties of a (p,q)-group.

The following lemma is an immediate consequence of Ito's Theorem [4, p. 434, Theorem 5.4]. So the proof will be omitted here.

Lemma 1. *Let p be a fixed prime in $\pi(G)$. Then G possesses a normal p-complement iff G contains no (p,q)-subgroup $\forall q \in \pi(G)$ with $q \neq p$.*

Theorem 1 is an immediate consequence of the following lemma:

Lemma 2. *Suppose that each subgroup of order p^2, where p is the smallest prime in $\pi(G)$, is normal in G. Then G has a normal p-complement or A_4 is involved in G.*

Proof. Let G be a counterexample. Then, by Lemma 1, G contains a (p,q)-subgroup K. Since A_4 is not involved in K, then $|P| = p^n$, where $P \in \mathrm{Syl}_p(K)$ and $n \geq 3$. Now, we shall make use of the properties of minimal nonnilpotent groups to be found in [3] (see also [4, p. 281, III.5.2]). Clearly P contains a subgroup H of order p^2. So the hypothesis of the lemma implies that $H \lhd G$ and consequently $H \lhd K$. If $P' = 1$, then K/H would be nilpotent. But then K would itself be nilpotent, a contradiction. Thus $P' \neq 1$. We have tha $P' = Z(P)$ is elementary abelian.

If $p = 2$, then $\mathrm{Exp}\, P = 4$. Hence there exists an element x of P of order 4 such that $x \notin P'$. The hypothesis of the lemma implies that $\langle x \rangle \lhd G$ and consequently $\langle x \rangle \lhd K$. Set $L = \langle x \rangle P'$. If $L = P$, then $P/P' \cong \langle x \rangle / \langle x \rangle \wedge P' = \langle x \rangle / \langle x \rangle \wedge Z(P)$. Now it follows that $|P/P'| = 2$ and this is impossible. Thus $L < P$. Obviously, $L \lhd K$. Now the structure of K yields K/L must be nilpotent. But then K would itself be nilpotent, a contradiction.

Now, we may assume that $p \neq 2$. $\mathrm{Exp}\, P = p$ and $P' \neq 1$. Then P contains an element $x \notin P'$. It is clear that $|x| = p$. Then x lies in a subgroup H of P of order p^2. Hence $H \not\leq P'$. If $L = P$, then $|P/P'| = p$ and this is impossible. Thus $L < P$. Now the structure of K yields K/L must be nilpotent. But then K would itself be nilpotent, a contradiction.

Theorem 2. *If G is an $\mathscr{H}_2$-N group and A_4 is not involved in G, then G is supersolvable.*

Proof. Let $S \in \mathrm{Syl}_2(G)$. Suppose that $|S|=2$. Then G has a normal 2-complement (Theorem 4.3 of [2, p. 252]). Now Theorem 2.2 of [2, p. 224] implies that for each prime $p \neq 2$, S leaves invariant some p-Sylow subgroup of G. Hence there exists a Hall subgroup H of order $2p^n$, where $p \in \pi(G)$. Theorem 7.2.15 of [5, p. 158] implies that H is supersolvable. Then H possesses a subgroup $H_1 = S\langle y\rangle$ of order $2p$. Since G is an $\mathscr{W}_2$-N group, $H_1 \lhd G$. Now Frattini argument yields $G = \langle y\rangle N_G(S)$. Set $N = N_G(S)$. Once again Theorem 7.2.15 of [5, p. 158] implies that we may assume $\pi(N)$ contains an odd prime $r \neq p$. Let $z \in N$ such that $|z| = r$. Since G is an $\mathscr{W}_2$-N group, $\langle z\rangle S \lhd G$. Since $S \operatorname{char} \langle z\rangle S \lhd G$, $S \lhd G$. Let x be an arbitrary element of G of odd prime order. Since G is an $\mathscr{W}_2$-N group, $\langle x\rangle S \lhd G$. Since $\langle x\rangle \operatorname{char} \langle x\rangle S \lhd G$, $\langle x\rangle \lhd G$. Thus G is a PN-1 group. Now Theorem 2 of [6] implies that G is supersolvable.

Now we may assume that $|S| = 2^n$, where $n \geq 2$. Let x be an element of order 2. Then x lies in a subgroup H of G of order 4. Since G is an $\mathscr{W}_2$-N group, $H \lhd G$. Let y be an element of odd prime order. Since A_4 is not involved in G, $[H, y] = 1$ and $[x, y] = 1$. Since $\langle x\rangle \operatorname{char} \langle x\rangle\langle y\rangle \lhd G$ and $\langle y\rangle \operatorname{char} \langle x\rangle\langle y\rangle \lhd G$, it follows that $\langle x\rangle \lhd G$ and $\langle y\rangle \lhd G$. Thus, G is a PN-1 group. We conclude therefore from Theorem 2 of [6] and Lemma 2 that G is supersolvable.

The proof of our next result is similar to that of Theorem 2. So the proof will be omitted here.

Theorem 3. *If G is an $\mathscr{W}_2$-N group and $q \equiv 1 \pmod p$ for some prime $q \in \pi(G)$, then G is supersolvable.*

Theorem 4. *If G is an $\mathscr{W}_2$-N group, then G' is nilpotent.*

Proof. Let S be a 2-Sylow subgroup of G. If $|S| = 2$, then G is supersolvable (see first paragraph of proof of Theorem 2). Now Theorem 9.1 of [4, p. 716] implies that G' is nilpotent. Once again Theorem 2 implies that $3 \mid |G|$.

Now we may assume that $|S| = 2^n$, where $n \geq 2$. Either $|\pi(G)| \geq 3$ or $|\pi(G)| = 2$. If $|\pi(G)| \geq 3$, let y be an element of prime order p, where $p \neq 2$ and $p \neq 3$. Let x be an element of order 2. Then x lies in a subgroup H of order 4. Since G is an $\mathscr{W}_2$-N group, $H \lhd G$. Hence $[H, y] = 1$ and $[x, y] = 1$. Since $\langle x\rangle \operatorname{char} \langle x\rangle\langle y\rangle \lhd G$ and $\langle y\rangle \operatorname{char} \langle x\rangle\langle y\rangle \lhd G$, $\langle x\rangle \lhd G$ and $\langle y\rangle \lhd G$. Let z be an element of order 3. Since $\langle x\rangle\langle z\rangle \lhd G$ and $\langle z\rangle \operatorname{char} \langle x\rangle\langle z\rangle$, it follows that $\langle z\rangle \lhd G$. It is clear that $G' \leq C_G(\langle x\rangle)$ for any element x of order 4. Now we have $G' \leq C_G(\langle x\rangle)$ for any element x of order 4 or a prime. By Theorem 5.5 of [4, p. 435], it follows that G' is nilpotent. If $|\pi(G)| = 2$, then $|G| = 2^n 3^m$. Hence G is solvable (Theorem 7.3 of [4, p. 492]). Let L be a minimal normal subgroup of G. Clearly L is elementary abelian. Suppose that $|L| = 3^b$. Let x be an element of order 2. Then x lies in a normal subgroup H of order 4. Let y be an element of L of order 3. Then $[H, L] = 1$ and $[x, y] = 1$. Since $\langle x\rangle \operatorname{char} \langle x\rangle\langle y\rangle \lhd G$, $\langle x\rangle \lhd G$. Let z be an element of G of order 3.

Since $\langle z\rangle$ char$\langle x\rangle\langle z\rangle \lhd G$, $\langle z\rangle \lhd G$. Now, we have $G' \le C_G(\langle x\rangle)$ for any element x of order 4, 2 or 3 and consequently G' is nilpotent. Hence, we may assume that $|L| = 2^a$. Since G is an $\mathscr{H}_2$-N group and L i a minimal normal subgroup, $|L| = 2$ or 4. If $|L| = 2$, then $G' \le C_G(\langle x\rangle)$ for every element x or order 4, 2 or 3 and consequently G' is nilpotent. Assume that $|L| = 4$. We argue that L is the 2-Sylow subgroup of G. Suppose false. Then G contains a subgroup $K > L$ of order 2^3. Since L is elementary abelian, K contains a maximal subgroup $L_1 \ne L$. Since G is an $\mathscr{H}_2$-group, $L_1 \lhd G$. But now $L_1 \wedge L$ is a normal subgroup of G of order 2, contradicting the minimality of L. Thus L is the 2-Sylow subgroup of G. If $3 \mid |C_G(L)|$, then $G' \le C_G(\langle x\rangle)$ for every element x of order 4, 2 and 3 and consequently G' is nilpotent. Hence $L = C_G(L)$. Now it follows easily that $G \cong A_4$ and consequently G' is nilpotent.

Theorem 5. *If G is an $\mathscr{H}_2$-N group, then G' is nilpotent.*

Proof. Let P be a p-Sylow subgroup of G, where p is the smallest odd prime in $\pi(G)$. Suppose that $|P| = p$. Set $N = N_G(P)$. If $P < N$, then N contains an element y of prime order $q \ne p$. Since G is an $\mathscr{H}_2$-N group, $\langle y\rangle P \lhd G$. Since P char $P\langle y\rangle \lhd G$, $P \lhd G$. Now, it follows that G is a PN-1 group. By Theorem 5.3 of [4, p. 283], G' is nilpotent. Hence we may assume that $P = N$. Thus G is a Frobenius group with complement P and kernel K. Theorem 3.1 of [2, p. 339] implies that K is nilpotent. Now, it follows easily that G' is nilpotent.

Suppose that $|P| = p^n$, where $n \ge 2$. Let x be an element of order p. Then x lies in a normal subgroup H of order p^2. Let y be an element of prime order $q \ne p$. Theorem 4.3 of [2, p. 252] implies that $[H, y] = 1$ and $[x, y] = 1$. Since $\langle x\rangle$ char$\langle x\rangle\langle y\rangle \lhd G$ and $\langle y\rangle$ char$\langle x\rangle\langle y\rangle \lhd G$, it follows that $\langle x\rangle \lhd G$ and $\langle y\rangle \lhd G$. Thus G is a PN-1 group. Once again Theorem 5.3 of [4, p. 283] implies that G' is nilpotent.

In [1], Narasimha and Deskins proved that if G is a PN-2 group, then G' is nilpotent. Theorems 4 and 5 generalize this result.

Lemma 3. *If $\mathscr{H}_3$ is empty, then $|G|$ is the product of at most three primes not necessarily distinct.*

Proof. Assume that G is not a 2-group. It is clear that if G is a 2-group, then $|G| \le 2^3$. Let S be a 2-Sylow subgroup of G. Since $\mathscr{H}_3$ is empty, $|S| \le 4$.

If $|S| = 2$, then $G = SK$, where K is a normal subgroup of G of odd order (Theorem 4.3 of [2, p. 252]). Theorem 2.2 of [2, p. 224] implies that for each prime $p \ne 2$, S leaves invariant some p-Sylow subgroup of G. Hence if there exists a p-Sylow subgroup P of G of order p^n, where $n \ge 2$, then G contains a Hall subgroup L of order $2p^n$. Theorem 7.2.15 of [5, p. 158] implies that L is supersolvable. Theorem 1 of [7, p. 279] implies that L contains a subgroup L_1 of order $2p^2$. Since $\mathscr{H}_3$ is empty, $|G| = |L_1| = 2p^2$. Now we may assume that K is of square free order. It follows easily that $|G| = 2p$ or $2pq$, where $2, p$ and q are distinct primes.

Suppose that $|S|=4$. Since $\mathscr{N}_3$ is empty, $N_G(S)=C_G(S)=S$. Thus, $G=SK$, where K is a normal subgroup of G of odd order. It is clear that G contains a subgroup L of order $2|K|$. The preceding paragraph implies that $|L|=2p$ or $2pq$, where 2, p and q are distinct primes. Since $\mathscr{N}_3$ is empty and $L<G$, $|L|=2p$ and consequently $|G|=2^2p$.

Theorem 6. *If G is n* $\mathscr{N}_3$-N *group, then* G' *is nilpotent.*

Proof. Let G be a counterexample. Let S be a 2-Sylow subgroup of G. Set $|S|=2^n$, where $n\geq 1$.

Case 1. Suppose that $n=1$. By Theorem 4.3 of [2, p. 252], G has a normal 2-complement and so $2\nmid|G'|$. If $S\lhd G$, then G/S is an $\mathscr{N}_2$-N group. By Theorem 5 $(G/S)'=G'S/S\cong G'/G'\wedge S=G'$ is nilpotent, a contradiction. Thus, S is not normal subgroup of G. It is very well known that if G is of square free order, then G is supersolvable, and consequently G' is nilpotent. Since G' is not nilpotent, so G contains a p-Sylow subgroup P of order p^m, where $m\geq 2$ and $p\neq 2$. Since G has a normal 2-complement, Theorem 2.2 of [2, p. 224] implies that for each prime $p\neq 2$, S leaves invariant some p-Sylow subgroup of G. Thus there exists a Hall subgroup H of order $2p^m$, where $m\geq 2$. Now, Theorem 7.2.15 of [5, p. 158] implies that H is supersolvable. Then H possesses a subgroup L of order $2p^2$ [7, Theorem 1, p. 279]. Let P_1 be a p-Sylow subgroup of L. Since G is an $\mathscr{N}_3$-N group, $L\lhd G$. Now Frattini's argument yields $G=P_1N_G(S)$. Let r be an odd prime such that $r\,|\,|N_G(S)|$. If $r^2\,|\,|N_G(S)|$, then $N_G(S)$ contains a subgroup M of order $2r^2$. Since G is an $\mathscr{N}_3$-N group, $M\lhd G$ and consequently $S\lhd G$, a contradiction. Thus $N_G(S)$ is of square free order. If $|\pi(N_G(S))|\geq 3$, then $N_G(S)$ contains a subgroup K of order $2r_1r_2$, where 2, r_1 and r_2 are distinct primes. Since G is an $\mathscr{N}_3$-N group, $K\lhd G$, a contradiction. Thus $|\pi(N_G(S))|\leq 2$. Since G' is not nilpotent and $|\pi(N_G(S))|\leq 2$, we have $|N_G(S)|=2r$, where 2, r and p are distinct primes. Now, it follows easily that $|G|=2rp^2$ and consequently G' is nilpotent, contradiction.

Case 2. Suppose that $n=2$. We argue that S is not a normal subgroup of G. Suppose false. Then Theorem 2.1 of [2, p. 221] implies that there exists a 2-complement K of G, $K\cong G/S$. It is clear that K is a PN-1 roup. Now Theorem 5.3 of [4, p. 283] implies that $(G/S)'=G'S/S\cong G'/G'\wedge S$ is nilpotent. Let x be an element of G of order a prime p, where $p\neq 2$ and $p\neq 3$. Theorem 4.3 of [2, p. 252] implies that $S\langle x\rangle=S\times\langle x\rangle$. Since G is an $\mathscr{N}_3$-N group, $S\langle x\rangle\lhd G$ and consequently $\langle x\rangle\lhd G$. But now $G/\langle x\rangle$ is an $\mathscr{N}_2$-N group. Theorem 4 implies that $(G/\langle x\rangle)'=G'\langle x\rangle/\langle x\rangle\cong G'/G'\wedge\langle x\rangle$ is nilpotent. Since G' is not nilpotent and $\langle x\rangle=p$, $G'/G'\wedge\langle x\rangle=G'/\langle x\rangle$. But $G'/(G'\wedge S)\wedge\langle x\rangle\subseteq G'/G'\wedge S\times G'/\langle x\rangle$, so G' is nilpotent, a contradiction. Hence $\pi(G)=\{2,3\}$. If $C_G(S)=G$, then $S\leq Z(G)$ and consequently G is nilpotent, a contradiction. Thus $C_G(S)<G$. Let x be an element of $C_G(S)$ of order 3. Then, $S\langle x\rangle=S\times\langle x\rangle$. Since G is an $\mathscr{N}_3$-N group, $S\langle x\rangle\lhd G$ and consequently $\langle x\rangle\lhd G$. Since $G'/\langle x\rangle$ and $G'/G'\wedge S$ are nilpotent, G' is nilpotent, a contradiction. Hence $C_G(S)=S<G$. Clearly S is elementary abelian. Since $G/C_G(S)\subseteq\mathrm{Aut}(S)$,

$|\mathrm{Aut}(S)|=6$, $C_G(S)=S$ and $\pi(G)=\{2,3\}$, it follows that $G\cong A_4$ and consequently G' is nilpotent, a contradiction. Thus S is not a normal subgroup of G. Set $N=N_G(S)$. If $S<N$, then N contains an element x of order a prime $p\neq 2$. Since G is an $\mathscr{H}_3$-N group, $\langle x\rangle S \lhd G$. Since S char $S\langle x\rangle \lhd G$, $S\lhd G$, a contradiction. Hence $N=S=C_G(S)$. Now, Theorem 4.3 of [2, p. 252] implies that G has a normal 2-complement and consequently A_4 is not involved in G. Suppose that $S\wedge S^x\neq 1$ for some $x\in G-N$. Then $S\wedge S^x=\langle y\rangle$, where $|y|=2$. Set $N_1=N_G(\langle y\rangle)$. If $N_1=G$, then $\langle y\rangle \lhd G$. Since G is an $\mathscr{H}_3$-N group, $G/\langle y\rangle$ is an $\mathscr{H}_2$-group. Theorem 2 implies that $G/\langle y\rangle$ is supersolvable and consequently G is supersolvable. Now, Theorem 9.1 of [4, p. 716] implies that G' is nilpotent, a contradiction. Thus $\langle y\rangle$ is not a normal subgroup of G. It is clear that N_1 contains an odd prime r. Let z be an element of N_1 of order r. Obviously N_1 is solvable. Since $\langle y\rangle$ is not a normal subgroup and G is an $\mathscr{H}_3$-group, it follows that $r\,|\,|N_1|$. Since N_1 is solvable, N_1 contains a Hall subgroup $L=S_1\langle z\rangle$, where $|S_1|=4$ and $|z|=r$ (Theorem 4.1 of [2, p. 231]). Since G is an $\mathscr{H}_3$-N group, $L\lhd G$. Since G has a normal 2-complement, $\langle z\rangle$ char L. Since $\langle z\rangle$ char $L\lhd G$, $\langle z\rangle \lhd G$. But then $G/\langle z\rangle$ is an $\mathscr{H}_2$-N group. Now, Theorem 2 implies that $G/\langle z\rangle$ is supersolvable and consequently G is supersolvable. Hence G' is nilpotent, a contradiction. Thus $S\wedge S^x=1$ for each element $x\in G-N$. Now, it follows that G is a Frobenius group with complement S and kernel K. Theorem 3.1 of [2, p. 339] implies that K is abelian. Now, it follows easily that G' is abelian, a contradiction.

Case 3. Suppose that $n\geq 3$. Let G denote a counterexample of least possible order. Lemma 3 and our choice of G imply that each proper subgroup of G is solvable. Let L be a minimal normal subgroup of G. Now it follows easily that $L<G$ and L is an elementary abelian p-group for some prime p (Theorem 1.5 of [2, p. 17]). Suppose that $p\neq 2$. Let S_1 be a subgroup of S of order 2^3. Since G is an $\mathscr{H}_3$-N group, $S_1\lhd G$. Let $y\in L$. Let S_2 be a maximal subgroup of S_1. Since $[S_1,L]=1$, $[S_2,y]=1$. Since G is an $\mathscr{H}_3$-N group, $S_2\times\langle y\rangle\lhd G$. Since $\langle y\rangle$ char $\langle y\rangle S_2\lhd G$ and S_2 char $\langle y\rangle S_2\lhd G$, $\langle y\rangle\lhd G$ and $S_2\lhd G$. It is clear that G/S_2 is a PN-1 group. Theorem 5.7 of [4, p. 436] implies that

$$(G/S_2)'=G'S_2/S_2\cong G'/G'\wedge S_2=S^*/G'\wedge S_2\cdot K/G'\wedge S_2,$$

where $S^*/G'\wedge S_2$ is a normal 2-Sylow subgroup of $G'/G'\wedge S_2$ and $K/G'\wedge S_2$ is nilpotent subgroup of $G'/G'\wedge S_2$ of odd order. Now, it follows that G'/S^* is nilpotent. Since G is an $\mathscr{H}_3$-N group, it follows that $G/\langle y\rangle$ is an $\mathscr{H}_2$-N group. Theorem 4 implies that $(G/\langle y\rangle)'$ is nilpotent. Since G' is not nilpotent and $|y|=p$, $(G/\langle y\rangle)'=G'/\langle y\rangle$. Since G'/S^* and $G'/\langle y\rangle$ are nilpotent, G' is nilpotent, a contradiction. Thus we must have $p=2$. Since L is a minimal normal subgroup of G and G is an $\mathscr{H}_3$-N group, $|L|\leq 2^3$.

Subcase 1. Suppose that $|L|=2$. Then G/L is an $\mathscr{H}_2$-N group. Theorem 4 implies that $(G/L)'=G'L/L$ is nilpotent. Since G' is not nilpotent and $|L|=2$, it follows that G'/L is nilpotent. Since G'/L is nilpotent and $|L|=2$, it follows that G' is nilpotent, a contradiction.

Subcase 2. Suppose that $|L|=4$. Since G is an $\mathscr{N}_3$-N group, G/L is a PN-1 group. Theorem 5.7 of [4, p. 436] implies that

$$(G/L)'=G'L/L\cong G'/G'\wedge L=S^*/G'\wedge L\cdot M/G'\wedge L$$

where $S^*/G'L$ is a normal 2-Sylow subgroup of $G'/G'\wedge L$ and $M/G'\wedge L$ is a nilpotent subgroup of $G'/G'\wedge L$ of odd order. Now, it follows that G'/S^* is nilpotent. Let x be an element of G of order a prime p, where $p\neq 2$ and $p\neq 3$. Since G is an $\mathscr{N}_3$-group, $L\langle x\rangle\lhd G$. By Theorem 4.3 of [2, p. 252], $L\langle x\rangle=L\times\langle x\rangle$. Since $\langle x\rangle$ char$\langle x\rangle L\lhd G$, $\langle x\rangle\lhd G$. Since G is an $\mathscr{N}_3$-N group, $G/\langle x\rangle$ is an $\mathscr{N}_2$-N group. Theorem 4 implies that $(G/\langle x\rangle)'=G'\langle x\rangle/\langle x\rangle\cong G'/G'\wedge\langle x\rangle$ is nilpotent. Since $G'/G'\wedge\langle x\rangle=G'/\langle x\rangle$. Since G'/S^* and $G'/\langle x\rangle$ are nilpotent, G' is nilpotent, a contradiction. Thus, $\pi(G)=\{2,3\}$. Set $C=C_G(L)$. Suppose that $3\,|\,|C_G(L)|=|C|$. Then C contains an element y of order 3. Since $L\langle y\rangle=L\times\langle y\rangle\lhd G$, $\langle y\rangle\lhd G$. Now we have that $G'/\langle y\rangle$ and G'/S^* are nilpotent, so G' is nilpotent, a contradiction. Thus, $3\nmid|C|$. Since $G/C\subsetneq \mathrm{Aut}(L)$ and $3\nmid|C|$, it follows that $|G|=2^3\cdot 3$. Let S be a 2-Sylow subgroup of G. since G is an $\mathscr{N}_3$-N group, $S\lhd G$. Since $|G|=2^2\cdot 3$ and $S\lhd G$, G' is nilpotent, a contradiction.

Subcase 3. Suppose that $|L|=2^3$. We argue that L is a 2-Sylow subgroup of G. Suppose false. Then there exists a subgroup $L^*>L$ of order 2^4. Since L is elementary abelian, L^* contains a maximal subgroup $L_1\neq L$. Since G is an $\mathscr{N}_3$-N group, $L_1\lhd G$. It follows that $L_1\wedge L$ is a normal subgroup of G of order 4, contradicting the minimality of L. Thus L is a 2-Sylow subgroup of G. Set $C=C_G(L)$. Let y be an element of C of order a prime p, where $p\neq 2$. Let K be a maximal subgroup of L. Since $[L,y]=1$, $[K,y]=1$. Since G is an $\mathscr{N}_3$-N group, $K\langle y\rangle\lhd G$. Since K char $K\langle y\rangle\lhd G$, $K\lhd G$, contradicting the minimality of L. Thus $C=C_G(L)=L$. Since $G/C\subsetneq\mathrm{Aut}(L)\cong\mathrm{GL}(3,2)$ and $|\mathrm{GL}(3,2)|=168$ and $C=L$, it follows that $|G|=24$, 56 or 168. Since G' is not nilpotent, $|G|\neq 24$ or 56. Since G is an $\mathscr{N}_3$-N group and G' is not nilpotent, it follows that $|G|\neq 168$. This contradiction completes the proof of the theorem.

In [1], Narsimha and Deskins proved that if G is a PN-3 group, then G is solvable and $\mathrm{Feit}(G)\leq 3$.

For the proof of the next lemma, see [4, Theorem 8.27 (Dickson), p. 213].

Lemma 4. *Set $G\cong L_2(q)$, where q is an odd prime power and $q\equiv 3$, 5 (mod 8). Assume that $\mathscr{N}_4$ is empty. Then*

(1) *$G\cong L_2(5)\cong A_5$, or*

(2) *$G\cong L_2(p)$, where p is a prime such that $p-1$ and $p+1$ are products of at most 3 primes, $p\equiv 3$, 5 (mod 8) and $p^2-1\not\equiv 0$ (mod 5), or*

(3) *$G\cong L_2(q)$, where $q=3^{2n+1}$ such that $q-1$ and $q+1$ are products of at most 3 primes and $q\equiv 3$, 5 (mod 8).*

We shall prove the following result:

Theorem 7. *if G is n $\mathscr{W}_4$-N group, then one of the following holds:*
(i) *G is solvable, or*
(ii) *G is isomorphic to* (1) *or* (2) *or* (3) *in the statement of Lemma* 4.

Proof. Let G be a counterexample. Let S be a 2-Sylow subgroup of G. Set $|S|=2^n$, $n\geq 1$.

Case 1. Suppose that $n\geq 4$. Let H be a subgroup of S of order 2^4. Since G is an $\mathscr{W}_4$-N group, $H\lhd G$. If H is non-abelian, then $|Z(H)|=2$ or 4. Since G is an $\mathscr{W}_4$-N group, $G/Z(H)$ is either an $\mathscr{W}_3$-N group or an $\mathscr{W}_2$-N group. Now, Theorems 6 and 4 yield that $G/Z(H)$ is solvable and consequently G is solvable, a contradicton. Thus, H is non-abelian. If H is cyclic, let L be a subgroup of H of order 4. Since L char $H\lhd G$, $L\lhd G$. Since G/L is an $\mathscr{W}_2$-N group, G/L is solvable and consequently G is solvable, a contradiction. Thus, H is non-cyclic abelian. We argue that $S=H$. Suppose false. Then there exists a subgroup $S_1>H$ of order 2^5. Since S_1 is non-cyclic, there exists a maximal subgroup K of S_1 such that $H\neq K$. Since G is an $\mathscr{W}_4$-N group, $K\lhd G$.

Now, it follows easily that $H\wedge K$ is a normal subgroup of G of order 2^3. Hence, $G/H\wedge K$ is a PN-1 group. By Theorem 5.7 of [4, p. 436], $G/H\wedge K$ is solvalbe and consequently G is solvable, a contradiction. Thus $H=S$. Here we shall not make use of the Feit–Thompson Theorem [8]. It is clear that we have four types of non-isomorphic non-cyclic abelian groups:

$$(2^2, 2^2),\quad (2, 2^3),\quad (2, 2, 2^2)\quad \text{and}\ (2, 2, 2, 2).$$

Suppose that S is not elementary abelian. Then $\Omega_1(S)$ is elementary abelian of order 4 or 8. Clearly, $\Omega_1(S)\lhd G$. Since G is an $\mathscr{W}_4$-N group, $G/\Omega_1(S)$ is either an $\mathscr{W}_2$-N group or a PN-1 group. Thus, $G/\Omega_2(S)$ is solvable, and consequently G is solvable, a contradiction. Now, we may assume that S is elementary abelian. Let y be an element of $C_G(S)$ of prime odd order. Then $[S, y]=1$. let S_1 be a maximal subgroup of S. Then, $[S_1, y]=1$. Since G is an $\mathscr{W}_4$-N group, $S_1\langle y\rangle\lhd G$ and consequently, $S_1\lhd G$. Since G/S_1 is a PN-1 group, G/S_1 is solvable and consequently G is solvable, a contradiction. Thus, $C_G(S)=S$. Since $G/C_G(S)\subsetneq \mathrm{Aut}(S)$ and $|\mathrm{Aut}(S)|=2^6\times 3^2\times 5\times 7$ and $C_G(S)=S$, it follows that $|G|/|S|\,|\,3^2\times 5\times 7$. Now, it follows easily that G is solvable, a contradiction.

Case 2. Suppose that $n=3$. Set $N=N_G(S)$. If $S\lhd G$, then G/S is a PN-1 group. Hence, G/S is solvable and consequently G is solvable, a contradiction. Thus, $N<G$. If $S<N$, let y be an element of N of prime odd order. Since G is an $\mathscr{W}_4$-N group, $S\langle y\rangle\lhd G$ and consequently $S\lhd G$, a contradiction. Thus $N=S$. We argue that S_4 is not involved in G. Suppose false. Then, there exist subgroups $H>K$ such that $H/K\cong S_4$. If $3\nmid |K|$, then the Schur–Zassenhaus Theorem implies that $H=KL$, where $L\cong S_4$. Since G is an $\mathscr{W}_4$-N group, $L\lhd G$. Now Frattini's argument yields that $G=LN_G(S)=L\cong S_4$, a contradiction. Thus $3\,|\,|K|$. Let Q be a 3-Sylow subgroup of K. If $|Q|=3$, let L/K be a subgroup of H/K of order 2^3. It is clear that L has a normal 2-complement and consequently L contains a Hall subgroup

L_1 of order $2^3 3$. Let S_1 be a 2-Sylow subgroup of L_1. Let Q_1 be a 3-Sylow subgroup of L_1. Since G is an $\mathscr{H}_4$-group, $L_1 = S_1 Q_1 \lhd G$ and consequently $Q_1 \lhd G$. Since G/Q_1 is an $\mathscr{H}_3$-N group, G/Q_1 is solvable and consequently G is solvable, a contradiction. Thus $|Q| \neq 3$. Suppose that $|Q| = 3^2$. Let L/K be a subgroup of H/K of order 4. Then L contains a Hall subgroup L_1 of order $2^2 3^2$. Now it follows easily that $S_1 \lhd L_1$ or $Q_1 \lhd L_1$, where S_1 and Q_1 are 2-and 3-Sylow subgroups of L_1, respectively. Since G is an $\mathscr{H}_4$-N group, $L_1 \lhd G$. Hence either $S_1 \lhd G$ or $Q_1 \lhd G$. Thus G/S_1 is an $\mathscr{H}_2$-N group or G/Q_1 is an $\mathscr{H}_2$-N group. This is a contradiction. Now suppose that $|Q| = 3^n$, where $n \geq 3$. Let L/K be a subgroup of G/K of order 2. Since L has a normal 2-complement, L contains a Hall subgroup L_1 of order 23^n. Since L_1 is supersolvable, L_1 contains a subgroup L_2 of order 23^3. Let S_2 be a 2-Sylow subgroup of L_2. Let Q_2 be a 2-Sylow subgroup of L_2. Since G is an $\mathscr{H}_4$-N group, $L_2 \lhd G$ and consequently $Q_2 \lhd G$. Now, it follows that each element of G/Q_2 of order 2 is normal. This is a contradiction as S_4 contains a dihedral group of order 8. Thus S_4 is not involved in G. Now a Theorem of Glauberman [9] implies that G has a normal 2-complement, i.e. $G = SK$, where K is a normal subgroup of G of odd order. Let y be an element of S of order 2. Set $G_1 = \langle y \rangle K$. Suppose that q is a prime divisor of $|K|$ with multiplicity at least 3. It is clear that G_1 contains a Hall subgroup L_1 of order $2q^n$, where $n \geq 3$. Since L_1 is supersolvable, L_1 contains a subgroup L_2 of order $2q^3$. Since G is an $\mathscr{H}_4$-N group, $L_2 \lhd G$. Now Frattini's argument yields that if $G = L_2 N_G(\langle y \rangle)/\langle y \rangle$ is an $\mathscr{H}_3$-N group, then $N_G(\langle y \rangle)$ is solvable. Hence G is solvable, a contradiction. Thus each prime divisor of $|K|$ appears with multiplicity at most 2. Now by a very well known result in the literature it follows that K possesses a Sylow tower and consequently K is solvable. Since $G/K \cong S$, and K is solvable, G is solvable, a contradiction.

Case 3. Suppose that $n = 2$. If $S \lhd G$, then $G/S \cong K$ is a $\mathscr{H}_2$-N group. Now Theorem 5 implies that K is solvable, a contradiction. Thus $N_G(S) < G$. It follows from the proof of Case 2 that if G has a normal 2-complement, G is solvable. Thus G has not a normal 2-complement. Now Burnside's Theorem implies that $C_G(S) < N_G(S)$. Now it follows easily that $C_G(S) = S$. Let $O(G)$ be the largest normal subgroup of odd order in the group G. By Theorem 2.1 of [2, p. 421], $G/O(G)$ is isomorphic to $L_2(q)$, $q \equiv 3, 5 \pmod 8$. We argue that $O(G) = 1$. Suppose false. Let $L/O(G)$ be a subgroup of $G/O(G)$ of order 2. It is clear that L has a normal 2-complement. Hence if $q^3 \mid |O(G)|$ for some prime divisor q of $|O(G)|$, then L contains a Hall subgroup L_1 of order $2q^n$, where $n \geq 3$. Since L_1 is supersolvable, there exists a subgroup L_2 of L_1 of order $2q^3$. Since G is an $\mathscr{H}_4$-N group, $L_2 \lhd G$, contradicting the simplicity of $G/O(G)$. Thus each prime divisor of $|O(G)|$ appears with multiplicity at most 2. Hence $O(G)$ possesses a Sylow tower. Let P be a p-Sylow subgroup of $O(G)$, where p is the largest prime in $\pi(O(G))$. Then $P \lhd G$. Clearly $|P| = p$ or p^2. Hence G/P is an $\mathscr{H}_3$-N group or an $\mathscr{H}_2$-N group, a contradiction. Thus $O(G) = 1$. Now Lemma 4 yields that G is isomorphic to (1) or (2) or (3), a contradiction.

Case 4. Suppose that $n=1$. Then $G=SK$, where K is a normal subgroup of G of odd order. Suppose that there exists a prime divisor q of $|K|$ which appears with multiplicity at least 3. Hence G contains a subgroup L of order $2q^3$. Since G is an $\mathscr{N}_4$-4 group, $L \lhd G$. Let y be an element of L of order 2. Frattini's argument yields that $G=LN_G(\langle y\rangle)$. It is clear that $N=\langle y\rangle \times L_1$, where L_1 is a normal subgroup of N of odd order. If $\langle y\rangle$ is not a normal subgroup of G, then $|L_1|$ is the product of at most 2 primes and consequently G is solvable, a contradiction. Thus $\langle y\rangle = S \lhd G$. let Q be the q-Sylow subgroup of L. Then $Q \lhd G$. Let Q_1 be a subgroup of Q of order q^2. If Q is cyclic, then $Q_1 \lhd G$. Hence G/Q_1 is an $\mathscr{N}_2$-N group, a contradiction. Thus, Q is not cyclic. We argue that Q is the q-Sylow subgroup of G. Suppose false. Then there exists a subgroup $H > Q$ of order q^4. Since H is not cyclic, H contains a maximal subgroup Q_0 such that $Q_0 \neq Q$. Since G is an $\mathscr{N}_4$-N group, $\langle y\rangle Q_0 \lhd G$ and consequently $Q_0 \lhd G$. It follows that $Q_0 \wedge Q$ is a normal subgroup of G of order q^2. Thus $G/Q_0 \wedge Q$ is an $\mathscr{N}_2$-N group, a contradiction. Thus Q is a normal q-Sylow subgroup of G. If there exsts a prime divisor $r \neq q$ of $|K|$ which appears with multiplicity at least 3, then $R \lhd G$, where R is an r-Sylow subgroup of G of order r^3. Now it follows easily that SQR is a normal nilpotent subgroup of G. Let K be a subgroup of SQR of order $2rq^2$. Let Q_1 be the q-Sylow subgroup of K. Since G is an $\mathscr{N}_4$-N group, $Q_1 \lhd G$. But then G/Q_1 is an $\mathscr{N}_2$-N group, a contradiction. Thus q is the only prime divisor of $|K|$ appearing with multiplicity 3. Now the Schur–Zassenhaus Theorem implies that $K/Q \cong K_1$, where K_1 is a subgroup of K and $K=Q_1K_1$. Now, it follows that K_1 possesses a Sylow tower as each prime divisor of $|K_1|$ appears with multiplicity at most 2. Thus K_1 is solvable and consequently K is solvable, a contradiction. Therefore, each prime divisor of $|K|$ appears with multiplicity at most ?. Thus K is solvable, a final contradiction.

It was proved in Janko [10] that if G is a finite non-abelian simple group all of whose chains of subgroups have length at most 4, then G is isomorphic to $L_2(p)$ for some prime $p>3$. This result follows at once from Theorem 7.

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